A RAPID METHOD FOR CALCULATING THE LEAST SQUARES SOLUTION OF A POLYNOMIAL OF DEGREE NOT EXCEEDING THE FIFTH*

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ABSTRACT. The ordinary method for finding the least squares solution of a polynomial is very laborious. If, however, the values of the independent variable form an arithmetic series, a method can be devised to give the solution very rapidly. The method given in this paper is an improvement on the method of Birge and Shea. The process of calculating the solution has been divided into three distinct parts :

(1) Calculating the solution corresponding to the standard n values, $x_r = r - \frac{1}{2}(n+1)$ of the independent variable. Tables are given for $n \leq 30$ to facilitate rapid calculation, and a powerful method has been shown for $n > 30$.

(2) Testing the suitability of the selected polynomial.

(3) Calculating the true from the standard solution.

The tables here given are more extensive than those of Birge and Shea, but every figure has been carefully tested on a calculating machine.

INTRODUCTION

The problem of fitting a number of observations to polynomials occurs frequently in physical sciences. An ideal illustration of the case is found in band spectra where it is known that most band series can be accurately represented by polynomials. Birge and Shea[1] have developed formulæ for fitting polynomials up to the fifth degree to data given for equally spaced values of the independent variable. But the methods by which they obtain the results are tentative throughout. A general and mathematically very elegant basis has been given to Birge and Shea's results by Condon.[2] But the general formula obtained by Condon for the coefficient of the highest power of the variable in the least squares solution for a polynomial is completely different from the value of the general coefficient obtained by Birge and Shea in its mathematical form. The transformation of the one general expression into the other has not yet been carried out. In the following pages I have obtained most of the equations of Birge and Shea by using Cramer's[3] rule for solving the normal equations, and have constructed tables which give the least squares solutions a little more rapidly than the corresponding tables given by Birge and Shea. Although the tables given here permit

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rapid caculations only if the number of observations does not exceed thirty, the general formulæ obtained here could be applied to a larger number of observations without very much increasing the labour.

§1. DERIVATION OF THE MAIN FORMULAE

It is supposed that n values of an observed quantity y are known corresponding to the n values of the independent variable x. It is assumed that the precision of measurement of x is large enough for the error in x to be negligible compared to the error in y. The individual x's and y's may be denoted by x_i, y $(i=1, 2,\ldots, n)$. The problem is to find constants $a_{\sigma\rho}$ such that if $f_\rho(x)=\sum\limits_{\sigma=0}^{\rho} a_{\sigma\rho}\, x^\sigma$, the sum of the squares of the residuals between the values of y calculated from this representation and the corresponding observed values of y is a minimum. The quantity M_ρ to be minimised is

$$M_\rho=\sum_{i=1}^{n}\left[y_i-f_\rho(x_i)\right]^2.$$

The conditions for a minimum are

$$\frac{\partial M_\rho}{\partial a_{\sigma\rho}}=0, \qquad \sigma=0,\ 1,\ 2,\ldots,\ \rho.$$

These $(\rho+1)$ equations are the normal equations for the $(\rho+1)$ unknowns $a_{\sigma\rho}$. Written out more fully, the normal equations assume the form

$$\sum_{i=1}^{n}\left\{y_i x_i^\sigma - a_{0\rho}x_i^\sigma - a_{1\rho}x_i^{(\sigma+1)} - \ldots\ldots\ldots - a_{\sigma\rho}x_i^{(\sigma+\rho)}\right\}=0$$

$$\sigma=0,\ 1,\ 2,\ldots\ldots\ldots\ldots,\ \rho.$$

Writing

$$\xi_\nu=\sum_{i=1}^{n} x_i^\nu.$$

and

$$\eta_\nu=\sum_{i=1}^{n} y_i x_i^\nu,$$

the normal equations can be put into the simpler form :

$$a_{0\rho}\xi_\sigma + a_{1\rho}\xi_{\sigma+1} + \ldots\ldots\ldots + a_{\sigma\rho}\xi_{(\sigma+\rho)} = \eta_\sigma$$

$$(\sigma=0,\ 1,\ 2,\ldots\rho)$$

It is now assumed that the n values of x form an arithmetic series. The series assumed for the purposes of calculation is the one in which

$$x_i=i-\frac{n+1}{2}.$$

This series is preferred to any other because it yields comparatively simple values for the sums ξ_ν. And once the coefficients $a_{\sigma\rho}$ are calculated with the standard series, the calculation of the corresponding coefficients for any other arithmetic series does not present any difficulty. With the values of x given by the standard arithmetic series,

$$\xi_\nu = \left(-\frac{n-1}{2}\right)^\nu + \left(-\frac{n-3}{2}\right)^\nu + \ldots\ldots\ldots + \left(\frac{n-3}{2}\right)^\nu + \left(\frac{n-1}{2}\right)^\nu.$$

It follows that for odd values of ν, ξ_ν vanishes, while for even values of ν, ξ_ν can be expressed in terms of Bernoulli polynomials[4] thus :

$$\xi_\nu = \frac{2}{\nu+1} B_{(\nu+1)}\left(\frac{n+1}{2}\right).$$

If these considerations are to apply to polynomials of degree not exceeding the fifth, the maximum value ν can take is 10. The highest Bernoulli polynomial required is the eleventh. The first six odd Bernoulli polynomials are therefore sufficient for the purpose. They are :

$$B_1(x) = x - \tfrac{1}{2}$$

$$B_3(x) = x(x-1)\,(x-\tfrac{1}{2})$$

$$B_5(x) = x(x-1)\,(x-\tfrac{1}{2})\,(x^2 - x - \tfrac{1}{3})$$

$$B_7(x) = x(x-1)\,(x-\tfrac{1}{2})\,(x^4 - 2x^3 + x + \tfrac{1}{3})$$

$$B_9(x) = x(x-1)\,(x-\tfrac{1}{2})\,(x^6 - 3x^5 + x^4 + 3x^3 - \tfrac{1}{5}x^2 - \tfrac{9}{5}x - \tfrac{3}{5})$$

$$B_{11}(x) = x(x-1)\,(x-\tfrac{1}{2})\,(x^8 - 4x^7 + \tfrac{8}{3}x^6 + 6x^5 - \tfrac{19}{3}x^4 - 8x^3 + \tfrac{2}{3}x^2 + 5x + \tfrac{5}{3}).$$

The values of ξ_ν calculated from the above are :

$$\xi_0 = n, \qquad \xi_1 = 0,$$

$$\xi_2 = \frac{n(n^2-1)}{12}, \qquad \xi_3 = 0,$$

$$\xi_4 = \frac{n(n^2-1)\,(3n^2-7)}{240}, \qquad \xi_5 = 0,$$

$$\xi_6 = \frac{n(n^2-1)\,(3n^4-18n^2+31)}{1344}, \qquad \xi_7 = 0,$$

$$\xi_8 = \frac{n(n^2-1)\,(5n^6-55n^4+239n^2-381)}{11520}, \qquad \xi_9 = 0,$$

$$\xi_{10} = \frac{n(n^2-1)\,(3n^8-52n^6+410n^4-1636n^2+2555)}{33792}.$$

A. *The polynomial is of degree 0 :* $f_0(x)=a_{00}$.

The normal equation is

$$a_{00}\xi_0=\eta_0$$

$$\therefore \quad a_{00}=\frac{\eta_0}{n}. \qquad \ldots \quad (1)$$

B. *The polynomial is of degree 1 :* $f_1(x)=a_{01}+a_{1}\ x$.

The two normal equations are :

$$a_{01}\xi_0=\eta_0$$

and

$$a_{11}\xi_2=\eta_1$$

$$\therefore \quad a_{01}=\frac{\eta_0}{n}=a_{00} \qquad \ldots \quad 2(a)$$

and

$$a_{11}=\frac{\eta_1}{\xi_2}=\frac{12\eta_1}{n(n^2-1)}. \qquad \ldots \quad 2(b)$$

C. *The polynomial is of degree 2 :* $f_2(x)=a_{02}+a_{12}x+a_{22}x^2$.

$$a_{02}\xi_0+a_{22}\xi_2=\eta_0$$

$$a_{12}\xi_2=\eta_1$$

and

$$a_{02}\xi_2+a_{22}\xi_4=\eta_2.$$

Writing

$$\Delta_2=\begin{vmatrix}\xi_0 & \xi_2\\ \xi_2 & \xi_4\end{vmatrix}$$

the solutions can be written in the form

$$\frac{a_{02}}{\xi_2\eta_2-\xi_4\eta_0}=\frac{a_{22}}{\xi_2\eta_0-\xi_0\eta_2}=-\frac{1}{\Delta_2}$$

and

$$a_{12}=\frac{\eta_1}{\xi_2}.$$

But Δ_2 can be easily shown to be equal to $\dfrac{n^2(n^2-1)(n^2-4)}{180}$ as the ξ's have already been worked out.

$$\therefore \quad a_{02}=\frac{\xi_4}{\Delta_2}\eta_0-\frac{\xi_2}{\Delta_2}\eta_2=\frac{3(3n^2-7)}{4n(n^2-4)}\eta_0-\frac{15}{n(n^2-4)}\eta_2 \qquad \ldots \quad 3(a)$$

$$a_{12}=\frac{\eta_1}{\xi_2}=\frac{12}{n(n^2-1)}\eta_1=a_{11} \qquad \ldots \quad 3(b)$$

and

$$a_{22}=-\frac{\xi_2}{\Delta_2}\eta_0+\frac{\xi_0}{\Delta_2}\eta_2=-\frac{15}{n(n^2-4)}\eta_0+\frac{180}{n(n^2-1)(n^2-4)}\eta_2 \ldots \quad 3(c)$$

D. *The polynomial is of degree 3:* $f_3(x)=a_{03}+a_{13}x+a_{23}x^2+a_{33}x^3$.

The four normal equations can be separated into two pairs thus:

$$\left.\begin{aligned} a_{03}\xi_0+a_{23}\xi_2&=\eta_0,\\ a_{03}\xi_2+a_{23}\xi_4&=\eta_2,\end{aligned}\right\} \text{ and } \left.\begin{aligned} a_{13}\xi_2+a_{33}\xi_4&=\eta_1,\\ a_{13}\xi_4+a_{33}\xi_6&=\eta_3.\end{aligned}\right\}$$

The solution is:

$$a_{03}=a_{02} \qquad \dots \quad 4(a)$$

$$a_{23}=a_{22} \qquad \dots \quad 4(c)$$

$$a_{13}=\frac{\xi_6}{\Delta_3}\eta_1-\frac{\xi_4}{\Delta_3}\eta_3$$

and

$$a_{33}=-\frac{\xi_4}{\Delta_3}\eta_1+\frac{\xi_2}{\Delta_3}\eta_3$$

where Δ_3 denotes $\begin{vmatrix} \xi_2 & \xi_4 \\ \xi_4 & \xi_6 \end{vmatrix}$ and can be shown to be equal to

$$\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)}{33600};$$

so that finally

$$a_{13}=\frac{25(3n^4-18n^2+31)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_1-\frac{140(3n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_3 \qquad \dots \quad 4(b)$$

and

$$a_{33}=-\frac{140(3n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_1+\frac{2800}{n(n^2-1)(n^2-4)(n^2-9)}\eta_3 \qquad \dots \quad 4(d)$$

E. *The polynomial is of degree 4:* $f_4(x)=a_{04}+a_{14}x+a_{24}x^2+a_{34}x^3+a_{44}x^4$.

(δ) The five normal equations can be separated into two groups of three and two equations:

$$\left.\begin{aligned} a_{04}\xi_0+a_{24}\xi_2+a_{44}\xi_4&=\eta_0\\ a_{04}\xi_2+a_{24}\xi_4+a_{44}\xi_6&=\eta_2\\ a_{04}\xi_4+a_{24}\xi_6+a_{44}\xi_8&=\eta_4\end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_{14}\xi_2+a_{34}\xi_4&=\eta_1\\ a_{14}\xi_4+a_{34}\xi_6&=\eta_3\end{aligned}\right\}$$

The solution is:

$$a_{14}=a_{13} \qquad \dots \quad 5(b)$$

$$a_{34}=a_{33} \qquad \dots \quad 5(d)$$

$$a_{04}=\frac{(\xi_4\xi_8-\xi_6^2)\eta_0-(\xi_2\xi_8-\xi_4\xi_6)\eta_2+(\xi_2\xi_6-\xi_4^2)\eta_4}{\Delta_4}$$

$$a_{24}=\frac{-(\xi_2\xi_8-\xi_4\xi_6)\eta_0+(\xi_0\xi_8-\xi_4^2)\eta_2-(\xi_0\xi_6-\xi_2\xi_4)\eta_4}{\Delta_4}$$

and $$a_{44}=\frac{(\xi_2\xi_6-\xi_4^2)\eta_0-(\xi_0\xi_6-\xi_2\xi_4)\eta_2+(\xi_0\xi_4-\xi_2^2)\eta_4}{\Delta_4}$$

where Δ_4 denotes $$\begin{vmatrix} \xi_0 & \xi_2 & \xi_4 \\ \xi_2 & \xi_4 & \xi_6 \\ \xi_4 & \xi_6 & \xi_8 \end{vmatrix}$$

and can be shown to be equal in value to

$$\frac{n^3(n^2-1)^2(n^2-4)^2(n^2-9)(n^2-16)}{7938000}.$$

Now with some simple manipulations with their rows and columns the values of the following determinants can be worked out to be :

$$(\xi_4\xi_8-\xi_6^2)=\begin{vmatrix} \xi_4 & \xi_6 \\ \xi_6 & \xi_8 \end{vmatrix}=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(15n^4-230n^2+407)}{33868800}$$

$$(\xi_2\xi_8-\xi_4\xi_6)=\begin{vmatrix} \xi_2 & \xi_4 \\ \xi_6 & \xi_8 \end{vmatrix}=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(n^2-7)}{120960}$$

$$(\xi_2\xi_6-\xi_4^2)=\Delta_3=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)}{33600}$$

$$(\xi_0\xi_8-\xi_4^2)=\begin{vmatrix} \xi_0 & \xi_4 \\ \xi_4 & \xi_8 \end{vmatrix}=\frac{n^2(n^2-1)(n^2-4)(n^4-10n^2+29)}{3600}$$

$$(\xi_0\xi_6-\xi_2\xi_4)=\begin{vmatrix} \xi_0 & \xi_2 \\ \xi_4 & \xi_6 \end{vmatrix}=\frac{n^2(n^2-1)(n^2-4)(3n^2-13)}{2520}$$

and $$(\xi_0\xi_4-\xi_2^2)=\Delta_2=\frac{n^2(n^2-1)(n^2-4)}{180}.$$

These provide all the necessary material for completing the solution as follows:

$$a_{04}=\frac{15(15n^4-230n^2+407)}{64n(n^2-4)(n^2-16)}\eta_0-\frac{525(n^2-7)}{8n(n^2-4)(n^2-16)}\eta_2$$
$$+\frac{945}{4n(n^2-4)(n^2-16)}\eta_4 \quad \cdots \quad 5(a)$$

$$a_{\;4}=\frac{-525(n^2-7)}{8n(n^2-4)(n^2-16)}\eta_0+\frac{2205(n^4-10n^2+29)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_2$$
$$-\frac{3150(3n^2-13)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_4 \quad \cdots \quad 5(c)$$

$$a_{44}=\frac{945}{4n(n^2-4)(n^2-16)}\eta_0-\frac{3150(3n^2-13)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_2$$

$$+\frac{44100}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_4 \qquad \ldots \quad 5(e)$$

F. *The polynomial is of degree* 5 : $f_5(x)=a_{05}+a_{15}x+a_{25}x^2+a_{35}x^3+a_{45}x^4+a_{55}x_5$.

The six normal equations split into two equal groups.

$$\left.\begin{aligned} a_{05}\xi_0+a_{25}\xi_2+a_{45}\xi_4&=\eta_0\\ a_{05}\xi_2+a_{25}\xi_4+a_{45}\xi_6&=\eta_2\\ a_{05}\xi_4+a_{25}\xi_6+a_{45}\xi_8&=\eta_4 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a_{15}\xi_2+a_{35}\xi_4+a_{55}\xi_6&=\eta_1\\ a_{15}\xi_4+a_{35}\xi_6+a_{55}\xi_8&=\eta_3\\ a_{15}\xi_6+a_{35}\xi_8+a_{55}\xi_{10}&=\eta_5 \end{aligned}\right\}$$

On solving the equations

$$a_{05}=a_{04} \qquad \ldots \quad 6(a)$$

$$a_{25}=a_{24} \qquad \ldots \quad 6(c)$$

$$a_{45}=a_{44} \qquad \ldots \quad 6(e)$$

$$a_{15}=\frac{(\xi_6\xi_{10}-\xi_8^2)\eta_1-(\xi_4\xi_{10}-\xi_6\xi_8)\eta_3+(\xi_4\xi_8-\xi_6^2)\eta_5}{\Delta_5}$$

$$a_{35}=\frac{-(\xi_4\xi_{10}-\xi_6\xi_8)\eta_1+(\xi_2\xi_{10}-\xi_6^2)\eta_3-(\xi_2\xi_8-\xi_4\xi_6)\eta_5}{\Delta_5}$$

$$a_{55}=\frac{(\xi_4\xi_8-\xi_6^2)\eta_1-(\xi_2\xi_8-\xi_4\xi_6)\eta_3+(\xi_2\xi_6-\xi_4^2)\eta_5}{\Delta_5}$$

where Δ_5 denotes
$$\begin{vmatrix} \xi_2 & \xi_4 & \xi_6\\ \xi_4 & \xi_6 & \xi_8\\ \xi_6 & \xi_8 & \xi_{10} \end{vmatrix}$$

and can be shown to be equal to

$$\frac{n^3(n^2-1)^3(n^2-4)^2(n^2-9)^2(n^2-16)(n^2-25)}{23471078400}.$$

Further it can similarly be shown that

$$(\xi_6\xi_{10}-\xi_8^2)=\begin{vmatrix} \xi_6 & \xi_8\\ \xi_8 & \xi_{10} \end{vmatrix}$$

$$=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(25n^8-900n^6+10230n^4-37060n^2+46137)}{2554675200}$$

$$(\xi_4\xi_{10}-\xi_6\xi_8)=\begin{vmatrix}\xi_4 & \xi_6\\ \xi_8 & \xi_{10}\end{vmatrix}=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(3n^6-75n^4+541n^2-853)}{21288960}$$

$(\xi_4\xi_8-\xi_6^2)$ from previous work

$$=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(15n^4-230n^2+407)}{33868800}$$

$$(\xi_2\xi_{10}-\xi_6^2)=\begin{vmatrix}\xi_2 & \xi_6\\ \xi_6 & \xi_{10}\end{vmatrix}=\frac{n^2(n^2-1)^2(n^2-4)(n^2-9)(3n^4-46n^2+199)}{1241856}$$

$(\xi_2\xi_8-\xi_4\xi_6)$ from previous work $=\dfrac{n^2(n^2-1)^2(n^2-4)(n^2-9)(n^2-7)}{120960}$

and $(\xi_2\xi_6-\xi_4^2)=\Delta_3=\dfrac{n^2(n^2-1)^2(n^2-4)(n^2-9)}{33600}$.

The solution may now be completed :

$$a_{15}=\frac{147(25n^8-900n^6+10230n^4-37060n^2+46137)}{16\,n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1$$

$$-\frac{2205(3n^6-75n^4+541n^2-853)}{2n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_3$$

$$+\frac{693(15n^4-230n^2+407)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5 \qquad \ldots\ 6(b)$$

$$a_{35}=\frac{-2205(3n^6-75n^4+541n^2-853)}{2n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1$$

$$+\frac{18900(3n^4-46n^2+199)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_3$$

$$-\frac{194040(n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5 \qquad \ldots\ 6(c)$$

$$a_{55}=\frac{693(15n^4-230n^2+407)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1$$

$$-\frac{194040(n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_3$$

$$+\frac{698544}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5. \qquad \ldots\ 6(f)$$

The coefficients $a_{\sigma\rho}$ $(0\leq\sigma\leq\rho\leq5)$ have been all expressed in terms of n and the sums η_ν $(\nu\leq\rho)$. Thus tables could be constructed to give the values of the various factors multiplying η_ν for different values of n. All that would be necessary for a rapid calculation of $a_{\sigma\rho}$ would be the evaluation of the η_ν from the n observed values y_i. However, from the form of η_ν, *viz.* $\eta_\nu = \sum_{i=1}^{n} y_i x_i^\nu$, and from the symmetrical spacing of x_i, it is at once evident that when ν is odd the factor multiplying y_{r+1} is the negative of the factor multiplying y_{n-r}; and when ν is even the two factors are identical. Further the values of $a_{\sigma\rho}$ obtained above in terms of n and η_ν are seen to involve η_ν *either* with odd values alone of ν *or* with even values alone of ν. These two facts permit the transformation of the values of $a_{\sigma\rho}$ into the one or the other of the following two forms : $a_{\sigma\rho}=$

either $\qquad \delta_1(y_n+y_1)+\delta_2(y_{n-1}+y_2)+\ldots\ldots$to $[\frac{n+1}{2}]$ terms

or $\qquad \delta_1(y_n-y_1)+\delta_2(y_{n-1}-y_2)+\ldots\ldots$to $[\frac{n}{2}]$ terms

where the pair factors δ_1, δ_2,... are functions of n and $[\frac{n+1}{2}]$ represents the integral part of $\frac{n+1}{2}$.

Tables could, therefore, also be constructed to give the values of the various pair-factors occurring in the new expression for $a_{\sigma\rho}$ for different values of n. The tables at the end are such tables valid for values of n up to 30. For larger values of n, use of formulæ (1) to (6) and computation of η's is recommended. The main formulae have been collected together on the last page of the tables. The tables given here have been calculated by the method mentioned above and have been tested by at least one of the polynomials $y=1+x^\rho$ ($\rho\leq 5$). It is, therefore, believed that the tables are free from all errors of calculation. The tables given by Birge and Shea for $a_{\rho\rho}$ form only one half of the tables given here. After the figures were calculated and tested, they were compared with the tables of Birge and Shea. The figures were found to agree in every case.

Before proceeding further, it would be interesting to see how the results of Birge and Shea could be shown to be identical with those of Condon, at least as far as they hold for polynomials of degree up to the fifth.

Consider the equations 1, 2(b), 3(c), 4(d), 5(e) and 6(f). If $a_{\rho\rho}$ be supposed known, the six values of η_ν can be calculated in terms of $a_{\rho\rho}$ as follows from the six equations:

$$\eta_0 = na_{00}$$

$$\eta_1 = \frac{n(n^2-1)}{12}a_{11}$$

$$\eta_2 = \frac{n(n^2-1)}{12}a_{00} + \frac{n(n^2-1)(n^2-4)}{180}a_{22}$$

$$\eta_3 = \frac{n(n^2-1)(3n^2-7)}{240}a_{11} + \frac{n(n^2-1)(n^2-4)(n^2-9)}{2800}a_{33}$$

$$\eta_4 = \frac{n(n^2-1)(3n^2-7)}{240}a_{00} + \frac{n(n^2-1)(n^2-4)(3n^2-13)}{2520}a_{22} + \frac{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}{44100}a_{44}$$

$$\eta_5 = \frac{n(n^2-1)(3n^4-18n^2+31)}{1344}a_{11} + \frac{n(n^2-1)(n^2-4)(n^2-9)(n^2-7)}{10080}a_{33}$$
$$+\frac{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}{698544}a_{55}.$$

Substituting these values of η_ν in the formulæ for $a_{\sigma\rho}(\sigma<\rho)$ the results obtained are:

$$a_{01}=a_{00}$$

$$a_{12}=a_{11}\ ; \quad a_{02}=a_{00}-\frac{n^2-1}{12}a_{22}$$

$$a_{23}=a_{22}\ ; \quad a_{03}=a_{00}-\frac{n^2-1}{12}a_{22}\ ; \quad a_{13}=a_{11}-\frac{3n^2-7}{20}a_{33}$$

$$a_{34}=a_{33}\ ; \ a_{14}=a_{13}\ ; \ a_{04}=a_{00}-\frac{n^2-1}{12}a_{22}+\frac{3(n^2-1)(n^2-9)}{560}a_{44};$$

$$a_{24}=a_{22}-\frac{3n^2-13}{14}a_{44}$$

$$a_{05}=a_{04}\ ; \ a_{25}=a_{24}\ ; \ a_{45}=a_{44}\ ; \ a_{15}=a_{11}-\frac{3n^2-7}{20}a_{33}+\frac{15n^4-230n^2+407}{1008}a_{55};$$

$$a_{35}=a_{33}-\frac{5(n^2-7)}{18}a_{55}.$$

These results sum up in brief most of the results of Birge and Shea. Now the orthogonal polynomials defined by Condon are:

$$\phi_1(x)=1$$
$$\phi_2(x)=x$$
$$\phi_3(x)=x^2-\frac{\xi_2}{\xi_0}=x^2-\frac{n^2-1}{12}$$
$$\phi_4(x)=x^3-\frac{\xi_4}{\xi_2}x=x^3-\frac{3n^2-7}{20}x$$
$$\phi_5(x)=x^4+\frac{\xi_0\xi_6-\xi_2\xi_4}{\xi_2^2-\xi_0\xi_4}x^2+\frac{\xi_4^2-\xi_2\xi_6}{\xi_2^2-\xi_0\xi_4}=x^4-\frac{(3n^2-13)}{14}x^2+\frac{3(n^2-1)(n^2-9)}{560}$$
$$\phi_6(x)=x^5+\frac{\xi_2\xi_8-\xi_4\xi_6}{\xi_4^2-\xi_2\xi_6}x^3+\frac{\xi_6^2-\xi_4\xi_8}{\xi_4^2-\xi_2\xi_6}=x^5-\frac{5(n^2-7)}{18}x^3+\frac{15n^4-230n^2+407}{1008}x$$

If use be now made of the coefficients R_{kl} defined by Birge and Shea,

$$\phi_1(x)=R_{00}$$
$$\phi_2(x)=R_{11}x$$
$$\phi_3(x)=R_{22}x^2+R_{02}$$

$$\phi_4(x) = R_{33}x^3 + R_{13}x$$
$$\phi_5(x) = R_{44}x^4 + R_{24}x^2 + R_{04}$$
$$\phi_6(x) = R_{55}x^5 + R_{35}x^3 + R_{15}x.$$

The identification is thus complete.

§II. DIRECTIONS FOR FINDING THE LEAST SQUARES SOLUTION

Subtract from each observation any convenient constant quantity. Arrange the resulting observations as indicated in model form 1 modified to suit the problem in view. To find any coefficient of the polynomial multiply the numbers in the column (A − B) or (A + B) as shown in the tables by the corresponding pair-factors and divide the algebraic sum of the products by the denominator written against the pair-factors. If finally the quantity subtracted from all the observed y's be added on to the constant term of the polynomial, the least squares solution is completed except as regards transformation of the coefficients into values corresponding to the true arithmetic series of values taken by the argument x.

If, however, it be necessary to test whether the polynomial chosen is suitable, the calculations necessary for the transformation could be postponed with advantage till the trend of the residuals between the observed and the calculated values of y has been examined. This can be done as in model form 2(a) or 2(b) according as n is odd or even. Starting from the central differences calculated by means of the formulae given in the model forms, the values of all the desired calculated values of y could be rapidly built up by successive addition or subtraction in the various columns in order from left to right.

If the fit is found suitable, the final transformation to the true values of the variable may be carried out now. Let the true set of n values of the independent variable be given by $x'_r = x'_1 + (r-1)d$ whereas the standard values x_r are given by $x_r = \frac{-(n-1)}{2} + (r-1)$. On eliminating $(r-1)$ and writing m for $x_1 + \frac{n-1}{2}d$, which is the centre of the true set, $x_r d = x'_r - m$ is obtained. The two following transformations could now be successively applied

$$x'' = xd$$

and

$$x' = x'' + m.$$

If $b_{\sigma p}$ be the coefficients with x'' as the variable and $c_{\sigma p}$ be the coefficients with x' as the variable, the calculation of $b_{\sigma p}$ from $a_{\sigma p}$ is indicated in model form 3(a), and the calculation of $c_{\sigma p}$ from $b_{\sigma p}$ is indicated in model form 3b. However, the calculations performed under model form 3(b) can be more conveniently carried out by Horner's scheme.[5]

Model Form 1

a_{55}	a_{35}	a_{15}	A−B	A	B	A+B	a_{05}	a_{25}	a_{45}
Pair-factors			$y_{21}-y_1$	y_{21}	y_1	$y_{21}+y_1$	Pair-factors		
,,	,,	,,	$y_{20}-y_2$	y_{20}	y_2	$y_{20}+y_2$	,,	,,	,,
,,	,,	,,	$y_{19}-y_3$	y_{19}	y_3	$y_{19}+y_3$	,,	,,	,,
,,	,,	,,	$y_{18}-y_4$	y_{18}	y_4	$y_{18}+y_4$	,,	,,	,,
,,	,,	,,	$y_{17}-y_5$	y_{17}	y_5	$y_{17}+y_5$	,,	,,	,,
,,	,,	,,	$y_{16}-y_6$	y_{16}	y_6	$y_{16}+y_6$	,,	,,	,,
,,	,,	,,	$y_{15}-y_7$	y_{15}	y_7	$y_{15}+y_7$	,,	,,	,,
,,	,,	,,	$y_{14}-y_8$	y_{14}	y_8	$y_{14}+y_8$	,,	,,	,,
,,	,,	,,	$y_{13}-y_9$	y_{13}	y_9	$y_{13}+y_9$	,,	,,	,,
,,	,,	,,	$y_{12}-y_{10}$	y_{12}	y_{10}	$y_{12}+y_{10}$	,,	,,	,,
—	—	,,	—	—	y_{11}	y_{11}	,,	,,	,,
Algebraic sum of products							Algebraic sum of products		
Denominators							Denominators		
a_{55}	a_{35}	a_{15}					a_{05}	a_{25}	a_{45}

Valid for 21 observations for the fifth degree polynomial.

Model Form 3(a)

Transformation to $x''(=xd)$

b_{0p}	b_{1p}	b_{2p}	b_{3p}	b_{4p}	b_{5p}
a_{0p}	$\frac{a_{1p}}{d}$	$\frac{a_{2p}}{d^2}$	$\frac{a_{3p}}{d^3}$	$\frac{a_{4p}}{d^4}$	$\frac{a_{5p}}{d^5}$

Model Form 3(b)

Transformation to x' $(=x''+m)$

b_5	$-5mb_5$	$+10m^2b_5$	$-10m^3b_5$	$+5m^4b_5$	$-m^5b_5$
	b_4	$-4mb_4$	$+6m^2b_4$	$-4m^3b_4$	$+m^4b_4$
		b_3	$-3mb_3$	$+3m^2b_3$	$-m^3b_3$
			b_2	$-2mb_2$	$+m^2b_2$
				b_1	$-mb_1$
					b_0
c_5	c_4	c_3	c_2	c_1	c_0

The suffix denoting the degree of the polynomial is left out in $c_{\sigma p}$ and is written c_σ valid for a polynomial of degree 5.

Omit the first row and the first column for a polynomial of degree 4. Similar modifications may be introduced for polynomials of still lower degrees.

MODEL FORM 2(a) n is odd

x_i	Δ^5	Δ^4	Δ^3	Δ^2	Δ	$y_{calc.}$	$y_{obs.}$	$y_{calc.} - y_{obs.}$
$\frac{n-1}{2}$								
1		$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
	$120a_5$		$30a_5+12a_4+6a_3$		$a_5+a_4+a_3+a_2+a_1$			
0		$24a_4$		$2a_4+2a_2$		a_0	$y_{\frac{n+1}{2}}$	$y_{\frac{n+1}{2}} - a_0$
			$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
-1		$\downarrow$		$\downarrow$				
$\frac{1-n}{2}$								

MODEL FORM 2(b) n is even

x_i	Δ^5	Δ^4	Δ^3	Δ^2	Δ	$y_{calc.}$	$y_{obs.}$	$y_{calc.} - y_{obs.}$
$\frac{n-1}{2}$								
$\frac{3}{2}$		$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\frac{1}{2}$		$60a_5+24a_4$		$\frac{15}{2}a_5+5a_4+3a_3+2a_2$		$\frac{a_5}{32}+\frac{a_4}{16}+\frac{a_3}{8}+\frac{a_2}{4}+\frac{a_1}{2}+a_0$	$y_{\frac{n+2}{2}}$	$y_{\frac{n+2}{2}} - \left(\frac{a_5}{32}+\frac{a_4}{16}+\frac{a_3}{8}+\frac{a_2}{4}+\frac{a_1}{2}+a_0\right)$
	$120a_5$		$15a_5+6a_3$		$\frac{a_5}{16}+\frac{a_3}{4}+a_1$			
$-\frac{1}{2}$		$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\frac{1-n}{2}$								

a_σ is written for $a_{\sigma p}$ in the expressions for the central differences. The tables are valid for fifth degree polynomials, but by omitting the column or columns containing the higher differences, the tables can be modified for lower degree polynomials.

$n=1$

	y_1	Denominator
a_{00}	1	1

$n=2$

	y_1+y_2	Denominator	
a_0	1	2	a_{01}

	y_2-y_1	Denominator
a_{11}	1	1

$n=3$

	y_3+y_1	y_2	Denominator	
a_{00}	1	1	3	a_{01}
a_{02}	0	1	1	—
a_{22}	1	-2	2	—

	y_3-y_1	Denominator	
a_{11}	1	2	a_{12}

$n=4$

	y_4+y_1	y_3+y_2	Denominator	
a_{00}	1	1	4	a_{01}
a_{02}	-1	9	16	a_{03}
a_{22}	1	-1	4	a_{23}

	y_4-y_1	y_3-y_2	Denominator	
a_{11}	3	1	10	a_{12}
a_{13}	-1	27	24	—
a_{33}	1	-3	6	—

$n=5$

	y_5+y_1	y_4+y_2	y_3	Denominator	
a_{00}	1	1	1	5	a_{01}
a_{02}	−3	12	17	35	a_{03}
a_{22}	2	−1	−2	14	a_{23}
a_{04}	0	0	1	1	—
a_{24}	−1	16	−30	24	—
a_{44}	1	−4	6	24	—

$n=6$

	y_6+y_1	y_5+y_2	y_4+y_3	Denominator	
a_{00}	1	1	1	6	a_{01}
a_{02}	−3	7	12	32	a_{03}
a_{22}	5	−1	−4	56	a_{23}
a_{04}	3	−25	150	256	a_{05}
a_{24}	−5	39	−34	96	a_{25}
a_{44}	1	−3	2	48	a_{45}

$n=7$

	y_7+y_1	y_6+y_2	y_5+y_3	y_4	Denominator	
a_{00}	1	1	1	1	7	a_{01}
a_{02}	−2	3	6	7	21	a_{03}
a_{22}	5	0	−3	−4	84	a_{23}
a_{04}	5	−30	75	131	231	a_{05}
a_{24}	−13	67	−10	−70	264	a_{25}
a_{44}	−3	−7	1	6	264	a_{45}

	y_5-y_1	y_4-y_2	Denominator	
a_{11}	2	1	10	a_{12}
a_{13}	−1	8	12	a_{14}
a_{33}	1	−2	12	a_{34}

	y_6-y_1	y_5-y_2	y_4-y_3	Denominator	
a_{11}	5	3	1	35	a_{12}
a_{13}	−275	1249	652	3024	a_{14}
a_{33}	5	−7	−4	108	a_{34}
a_{15}	9	−125	2250	1920	—
a_{35}	−1	13	−34	48	—
a_{55}	1	−5	10	120	—

	y_7-y_1	y_6+y_2	y_5-y_3	Denominator	
a_{11}	3	2	1	28	a_{12}
a_{13}	−22	67	58	252	a_{14}
a_{33}	1	−1	−1	36	a_{34}
a_{15}	1	−9	45	60	—
a_{35}	−1	8	−13	48	—
a_{55}	1	−4	5	240	—

$n=8$

	y_8+y_1	y_7+y_2	y_6+y_3	y_5+y_4	Denominator	
a_{00}	1	1	1	1	8	a_{01}
a_{02}	−3	3	7	9	32	a_{03}
a_{22}	7	1	−3	−5	168	a_{23}
a_{04}	15	−69	85	225	512	a_{05}
a_{24}	−91	345	39	−293	2112	a_{25}
a_{44}	7	−13	−3	9	1056	a_{45}

$n=9$

	y_9+y_1	y_8+y_2	y_7+y_3	y_6+y_4	y_5	Denominator	
a_{00}	1	1	1	1	1	9	a_{01}
a_{02}	−21	14	39	54	59	231	a_{03}
a_{22}	28	7	−8	−17	−20	924	a_{23}
a_{04}	15	−55	30	135	179	429	a_{05}
a_{24}	−126	371	151	−211	−370	3432	a_{25}
a_{44}	14	−21	−11	9	18	3432	a_{45}

	y_8-y_1	y_7-y_2	y_6-y_3	y_5-y_4	Denominator	
a_{11}	7	5	3	1	84	a_{12}
a_{13}	−889	1955	2209	909	11088	a_{14}
a_{33}	7	−5	−7	−3	396	a_{34}
a_{15}	13573	−92437	249283	168445	549120	—
a_{35}	−245	1429	−1427	−1149	13728	—
a_{55}	7	−23	17	15	3120	—

	y_9-y_1	y_8-y_2	y_7-y_3	y_6-y_4	Denominator	
a_{11}	4	3	2	1	60	a_{12}
a_{13}	−86	142	193	126	1188	a_{14}
a_{33}	14	−7	−13	−9	1188	a_{34}
a_{15}	254	−1381	2269	2879	8580	—
a_{35}	−100	457	−256	−459	6864	—
a_{55}	4	−11	4	9	3120	—

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$n=10$

	$y_{10}+y_1$	y_9+y_2	y_8+y_3	y_7+y_4	y_6+y_5	Denominator	
a_{00}	1	1	1	1	1	10	a_{01}
a_{02}	−14	6	21	31	36	160	a_{03}
a_{22}	6	2	−1	−3	−4	264	a_{23}
a_{04}	10	−30	3	55	90	256	a_{05}
a_{24}	−426	1006	645	−279	−946	13728	a_{25}
a_{44}	18	−22	−17	3	18	6864	a_{45}

$n=11$

	$y_{11}+y_1$	$y_{10}+y_2$	y_9+y_3	y_8+y_4	y_7+y_5	y_6	Denominator	
a_{00}	1	1	1	1	1	1	11	a_{01}
a_{02}	−36	9	44	69	84	89	429	a_{03}
a_{22}	15	6	−1	−6	−9	−10	858	a_{23}
a_{04}	18	−45	−10	60	120	143	429	a_{05}
a_{24}	−90	174	146	1	−136	−190	3432	a_{25}
a_{44}	6	−6	−6	−1	4	6	3432	a_{45}

	$y_{10}-y_1$	y_9-y_2	y_8-y_3	y_7-y_4	y_6-y_5	Denominator	
a_{11}	9	7	5	3	1	165	a_{12}
a_{13}	−1338	1694	2675	2191	828	20592	a_{14}
a_{33}	42	−14	−35	−31	−12	5148	a_{34}
a_{15}	44274	−198506	200579	390769	188674	1372800	—
a_{35}	−402	1498	−347	−1457	−762	34320	—
a_{55}	6	−14	1	11	6	7800	—

	$y_{11}-y_1$	$y_{10}-y_2$	y_9-y_3	y_8-y_4	y_7-y_5	Denominator	
a_{11}	5	4	3	2	1	110	a_{12}
a_{13}	−300	294	532	503	296	5148	a_{14}
a_{33}	30	−6	−22	−23	−14	5148	a_{34}
a_{15}	573	−2166	1249	3774	3084	17160	—
a_{35}	−129	402	11	−340	−316	13728	—
a_{55}	3	−6	−1	4	4	6240	—

$n=12$

	$y_{12}+y_1$	$y_{11}+y_2$	$y_{10}+y_3$	y_9+y_4	y_8+y_5	y_7+y_6	Denominator	
a_{00}	1	1	1	1	1	1	12	a_{01}
a_{02}	−9	1	9	15	19	21	112	a_{03}
a_{22}	55	25	1	−17	−29	−35	4004	a_{23}
a_{04}	45	−95	−45	87	220	300	1024	a_{05}
a_{24}	−1221	1959	1989	545	−1116	−2156	54912	a_{25}
a_{44}	33	−27	−33	−13	12	28	27456	a_{45}

	$y_{12}-y_1$	$y_{11}-y_2$	$y_{10}-y_3$	y_9-y_4	y_8-y_5	y_7-y_6	Denominator	
a_{11}	11	9	7	5	3	1	286	a_{12}
a_{13}	−1617	1227	2541	2665	1939	703	30888	a_{14}
a_{33}	33	−3	−21	−25	−19	−7	7722	a_{34}
a_{15}	626307	−2028363	515361	3020151	3310276	1396860	18670080	—
a_{35}	−3531	9363	2247	−6367	−8516	−3772	466752	—
a_{55}	33	−57	−21	29	41	20	106080	—

$n=13$

	$y_{13}+y_1$	$y_{12}+y_2$	$y_{11}+y_3$	$y_{10}+y_4$	y_9+y_5	y_8+y_6	y_7	Denominator	
a_{00}	1	1	1	1	1	1	1	13	a_{01}
a_{02}	−11	0	9	16	21	24	25	143	a_{03}
a_{22}	22	11	2	−5	−10	−13	−14	2002	a_{23}
a_{04}	110	−198	−135	110	390	600	677	2431	a_{05}
a_{24}	−2211	2970	3504	1614	−971	−016	−3780	116688	a_{25}
a_{44}	99	−66	−96	−54	11	64	84	116688	a_{45}

	$y_{13}-y_1$	$y_{12}-y_2$	$y_{11}-y_3$	$y_{10}-y_4$	y_9-y_5	y_8-y_6	Denominator	
a_{11}	6	5	4	3	2	1	182	a_{12}
a_{13}	−1133	660	1578	1796	1489	832	24024	a_{14}
a_{33}	11	0	−6	−8	−7	−4	3432	a_{34}
a_{15}	9647	−27093	12	33511	45741	31380	291720	—
a_{35}	−1430	3267	1374	−1633	−3050	−2252	233376	—
a_{55}	22	−33	−18	11	26	20	106080	—

$n = 14$

	$y_{14}+y_1$	$y_{13}+y_2$	$y_{12}+y_3$	$y_{11}+y_4$	$y_{10}+y_5$	y_9+y_6	y_8+y_7	Denominator	
a_{00}	1	1	1	1	1	1	1	14	a_{01}
a_{02}	−33	−3	22	42	57	67	72	448	a_{03}
a_{22}	13	7	2	−2	−5	−7	−8	1456	a_{23}
a_{04}	165	−255	−220	60	417	725	900	3584	a_{05}
a_{24}	−7579	8560	11484	6916	−535	−7419	−11436	466752	a_{25}
a_{44}	143	−77	−132	−92	−13	63	108	233376	a_{45}

	$y_{14}-y_1$	$y_{13}-y_2$	$y_{12}-y_3$	$y_{11}-y_4$	$y_{10}-y_5$	y_9-y_6	y_8-y_7	Denominator	
a_{11}	13	11	9	7	5	3	1	455	a_{12}
e_{13}	−69641	30547	85998	104846	95225	65260	23112	1633632	a_{14}
a_{33}	143	11	−66	−98	−95	−67	−24	58344	a_{34}
a_{15}	5711277	−14039553	−29144428	13982052	23239761	20544275	8078100	177365760	—
a_{35}	−22165	44033	25476	−13916	−39329	−38587	−15684	4434144	—
a_{55}	143	−187	−132	28	139	145	60	1007760	—

$n = 15$

	$y_{15}+y_1$	$y_{14}+y_2$	$y_{13}+y_3$	$y_{12}+y_4$	$y_{11}+y_5$	$y_{10}+y_6$	y_9+y_7	y_8	Denominator	
a_{00}	1	1	1	1	1	1	1	1	15	a_{01}
a_{02}	−78	−13	42	87	122	147	162	167	1105	a_{03}
a_{22}	91	52	19	−8	−29	−44	−53	−56	12376	a_{23}
a_{04}	2145	−2860	−2937	−165	3755	7500	10125	11063	46189	a_{05}
a_{24}	−31031	29601	44495	31856	6579	−19751	−38859	−45780	2217072	a_{25}
a_{44}	1001	−429	−869	−704	−249	251	621	756	2217072	a_{45}

	$y_{15}-y_1$	$y_{14}-y_2$	$y_{13}-y_3$	$y_{12}-y_4$	$y_{11}-y_5$	$y_{10}-y_6$	y_9-y_7	Denominator	
a_{11}	7	6	5	4	3	2	1	280	a_{12}
a_{13}	−12922	4121	14150	18334	17842	13843	7506	334152	$a_{\;4}$
a_{33}	91	13	−35	−58	−61	−49	−27	47736	a_{34}
a_{16}	78351	−169819	−65229	130506	266401	279975	175425	2519400	—
a_{35}	−8281	14404	10379	−1916	−11671	−14180	−9315	2015520	—
a_{66}	1001	−1144	−979	−44	751	1000	675	10077600	—

$n = 19$

	$y_{19}+y_1$	$y_{18}+y_2$	$y_{17}+y_3$	$y_{16}+y_4$	$y_{15}+y_5$	$y_{14}+y_6$	$y_{13}+y_7$	$y_{12}+y_8$	$y_{11}+y_9$	y_{10}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	19	a_{01}
a_{02}	−136	−51	24	89	144	189	224	249	264	269	2261	a_{03}
a_{22}	51	34	19	6	−5	−14	−21	−26	−29	−30	13566	a_{23}
a_{04}	340	−255	−420	−290	18	405	790	1110	1320	1393	7429	a_{05}
a_{24}	−32028	15028	35148	36357	25610	8792	−9282	−24867	−35288	−38940	3922512	a_{25}
a_{44}	612	−68	−388	−453	−354	−168	42	227	352	396	3922512	a_{15}

	$y_{19}-y_1$	$y_{18}-y_2$	$y_{17}-y_3$	$y_{16}-y_4$	$y_{15}-y_5$	$y_{14}-y_6$	$y_{13}-y_7$	$y_{12}-y_8$	$y_{11}-y_9$	Denominator	
a_{11}	9	8	7	6	5	4	3	2	1	570	a_{12}
a_{13}	−6036	−68	4648	7481	8700	8574	7372	5363	2816	255816	a_{14}
a_{33}	204	68	−28	−89	−120	−126	−112	−83	−44	255816	a_{34}
a_{15}	255102	−349928	−322378	−9473	348823	604484	686099	583549	332684	9806280	—
a_{35}	−15810	16796	20342	9818	−4329	−15546	−20525	−18554	−10368	7845024	—
a_{55}	102	−68	−98	−58	3	54	79	74	44	3505920	—

$n = 20$

	$y_{20}+y_1$	$y_{19}+y_2$	$y_{18}+y_3$	$y_{17}+y_4$	$y_{16}+y_5$	$y_{15}+y_6$	$y_{14}+y_7$	$y_{13}+y_8$	$y_{12}+y_9$	$y_{11}+y_{10}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	20	a_{01}
a_{02}	−153	−63	17	87	147	197	237	267	287	297	2640	a_{03}
a_{22}	57	39	23	9	−3	−13	−21	−27	−31	−33	17556	a_{23}
a_{04}	1530	−990	−1802	−1410	−255	2285	2895	4323	5380	5940	33792	a_{05}
a_{24}	−226746	87006	231370	253826	195919	93259	−24179	−133555	−216164	−260436	31380096	a_{25}
a_{44}	1938	−102	−1122	−1402	−1187	−687	−77	503	948	1188	15690048	a_{35}

	$y_{20}-y_1$	$y_{19}-y_2$	$y_{18}-y_3$	$y_{17}-y_4$	$y_{16}-y_5$	$y_{15}-y_6$	$y_{14}-y_7$	$y_{13}-y_8$	$y_{12}-y_9$	$y_{11}-y_{10}$	Denominator	
a_{11}	19	17	15	13	11	9	7	5	3	1	1330	a_{12}
a_{13}	−1030047	−69819	606475	1032239	1240877	1265793	1140391	898075	572249	196317	41186376	a_{14}
a_{33}	969	357	−85	−377	−539	−591	−553	−445	−287	−99	1470942	a_{34}
a_{15}	388722102	−478825638	−495267358	−99299538	397343727	792062409	978962229	922926339	651687004	234397284	15690048000	—
a_{35}	−672486	634134	842894	486434	−53911	−527937	−796397	−810827	−593372	−216612	392251200	—
a_{55}	1938	−1122	−1802	−1222	−187	771	1351	1441	1076	396	89148000	—

$n=21$

	$y_{21}+y_1$	$y_{20}+y_2$	$y_{19}+y_3$	$y_{18}+y_4$	$y_{17}+y_5$	$y_{16}+y_6$	$y_{15}+y_7$	$y_{14}+y_8$	$y_{13}+y_9$	$y_{12}+y_{10}$	y_{11}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	21	a_{01}
a_{02}	−171	−76	9	84	149	204	249	284	309	324	329	3059	a_{03}
a_{22}	190	133	82	37	−2	−35	−62	−83	−98	−107	−110	67298	a_{23}
a_{04}	11628	−6460	−13005	−11220	−3940	6378	17655	28190	36660	42120	44003	260015	a_{05}
a_{24}	−12597	3876	11934	13804	11451	6578	626	−5226	−10061	−13224	−14322	1961256	a_{25}
a_{44}	969	0	−510	−680	−615	−406	−130	150	385	540	594	9806280	a_{45}

	$y_{21}-y_1$	$y_{20}-y_2$	$y_{19}-y_3$	$y_{18}-y_4$	$y_{17}-y_5$	$y_{16}-y_6$	$y_{15}-y_7$	$y_{14}-y_8$	$y_{13}-y_9$	$y_{12}-y_{10}$	Denominator	
a_{11}	10	9	8	7	6	5	4	3	2	1	770	a_{12}
a_{13}	−84075	−10032	43284	78176	96947	101900	95338	79564	56881	29592	3634092	a_{14}
a_{33}	285	114	−12	−98	−149	−170	−166	−142	−103	−54	519156	a_{34}
a_{15}	15033066	−16649358	−19052988	−6402438	10949942	26040033	34807914	35613829	28754154	15977364	637408200	—
a_{35}	−748068	625974	908004	598094	62644	−448909	−787382	−887137	−749372	−425412	509926560	—
a_{55}	3876	−1938	−3468	−2618	−788	1063	2354	2819	2444	1404	231784800	—

$n=22$

	$y_{22}+y_1$	$y_{21}+y_2$	$y_{20}+y_3$	$y_{19}+y_4$	$y_{18}+y_5$	$y_{17}+y_6$	$y_{16}+y_7$	$y_{15}+y_8$	$y_{14}+y_9$	$y_{13}+y_{10}$	$y_{12}+y_{11}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	22	a_{01}
a_{02}	−19	−9	0	8	15	21	26	30	33	35	36	352	a_{03}
a_{22}	35	25	16	8	1	−5	−10	−14	−17	−19	−20	14168	a_{23}
a_{04}	1615	−765	−1710	−1598	−765	495	1930	3330	4527	5395	5850	36608	a_{05}
a_{24}	−171969	41211	151050	183450	161435	105151	31866	−44030	−111025	−160485	−186654	29995680	a_{25}
a_{44}	1197	57	−570	−810	−775	−563	−258	70	365	585	702	14997840	a_{45}

	$y_{22}-y_1$	$y_{21}-y_2$	$y_{20}-y_3$	$y_{19}-y_4$	$y_{18}-y_5$	$y_{17}-y_6$	$y_{16}-y_7$	$y_{15}-y_8$	$y_{14}-y_9$	$y_{13}-y_{10}$	$y_{12}-y_{11}$	Denominator	
a_{11}	21	19	17	15	13	11	9	7	5	3	1	1771	u_{12}
a_{13}	−173299	−28671	77520	149320	190775	205931	198834	173530	134065	84485	28836	8075760	a_{14}
a_{33}	133	57	0	−40	−65	−77	−78	−70	−55	−35	−12	288420	a_{34}
a_{15}	686844319	−684364971	−857795622	−383761162	328835403	995865057	1441336242	1580758326	1404505071	961178101	340970370	30596693600	—
a_{35}	−965447	716091	1129854	825554	214149	−415833	−881970	−1091622	−1025607	−721877	−259194	764889840	—
a_{55}	2261	−969	−1938	−1598	−663	363	1158	1554	1509	1079	390	173838600	—

$n=23$

	$y_{23}+y_1$	$y_{22}+y_2$	$y_{21}+y_3$	$y_{20}+y_4$	$y_{19}+y_5$	$y_{18}+y_6$	$y_{17}+y_7$	$y_{16}+y_8$	$y_{15}+y_9$	$y_{14}+y_{10}$	$y_{13}+y_{11}$	y_{12}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	23	a_{01}
a_{02}	−42	−21	−2	15	30	43	54	63	70	75	78	79	805	a_{03}
a_{22}	77	56	37	20	5	−8	−19	−28	−35	−40	−43	−44	35420	a_{23}
a_{04}	95	−38	−95	−95	−55	10	87	165	235	290	325	337	2185	a_{05}
a_{24}	−115577	20615	93993	119510	110545	78903	34815	−13062	−57645	−93425	−116467	−124410	22496760	a_{25}
a_{44}	1463	133	−627	−950	−955	−747	−417	−42	315	605	793	858	22496760	a_{45}

	$y_{23}-y_1$	$y_{22}-y_2$	$y_{21}-y_3$	$y_{20}-y_4$	$y_{19}-y_5$	$y_{18}-y_6$	$y_{17}-y_7$	$y_{16}-y_8$	$y_{15}-y_9$	$y_{14}-y_{10}$	$y_{13}-y_{11}$	Denominator	
a_{11}	11	10	9	8	7	6	5	4	3	2	1	1012	a_{12}
a_{13}	−3938	−815	1518	3140	4130	4567	4530	4098	3350	2365	1222	197340	a_{14}
a_{33}	77	35	3	−20	−35	−43	−45	−42	−35	−25	−13	197340	a_{34}
a_{15}	400653	−359157	−489687	−265164	106911	478349	752859	878634	840937	654687	357045	18747300	—
a_{35}	−49115	32224	55233	43928	16583	−13632	−38013	−51684	−52959	−42704	−23699	44993520	—
a_{55}	209	−76	−171	−152	−77	12	87	132	141	116	65	20451600	—

$n=24$

	$y_{24}+y_1$	$y_{23}+y_2$	$y_{22}+y_3$	$y_{21}+y_4$	$y_{20}+y_5$	$y_{19}+y_6$	$y_{18}+y_7$	$y_{17}+y_8$	$y_{16}+y_9$	$y_{15}+y_{10}$	$y_{14}+y_{11}$	$y_{13}+y_{12}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	24	a_{01}
a_{02}	−231	−121	−21	69	149	219	279	329	369	399	419	429	4576	a_{03}
a_{22}	253	187	127	73	25	−17	−53	−83	−107	−125	−137	−143	131560	a_{23}
a_{04}	3135	−1045	−2955	−3135	−2071	−195	2115	4535	6795	8679	10025	10725	73216	a_{05}
a_{24}	−43769	5379	32909	43721	42225	32341	17499	639	−15789	−29825	−39999	−45331	9472320	a_{25}
a_{44}	253	33	−97	−157	−165	−137	−87	−27	33	85	123	143	4736160	a_{45}

	$y_{24}-y_1$	$y_{23}-y_2$	$y_{22}-y_3$	$y_{21}-y_4$	$y_{20}-y_5$	$y_{19}-y_6$	$y_{18}-y_7$	$y_{17}-y_8$	$y_{16}-y_9$	$y_{15}-y_{10}$	$y_{14}-y_{11}$	$y_{13}-y_{12}$	Denominator	
a_{11}	23	21	19	17	15	13	11	9	7	5	3	1	2300	a_{12}
a_{13}	−396451	−96943	130283	292111	395425	447109	454047	423123	361221	275225	172019	58487	21312720	a_{14}
a_{33}	1771	847	133	−391	−745	−949	−1023	−987	−861	−665	−419	−143	5328180	a_{34}
a_{15}	2124400773	−1712322097	−2528019597	−1574705427	184892971	2016887349	3585776189	4373993139	4451455449	3795112407	2538493775	891258225	104384966400	−
a_{35}	−2475605	1432277	2668189	2279411	1075381	−347269	−1579677	−2375715	−2626953	−2337623	−1599583	−567281	2609624160	−
a_{55}	4807	−1463	−3743	−3553	−2071	−169	1551	2721	3171	2893	2005	715	593096400	−

$n=25$

	$y_{25}+y_1$	$y_{24}+y_2$	$y_{23}+y_3$	$y_{22}+y_4$	$y_{21}+y_5$	$y_{20}+y_6$	$y_{19}+y_7$	$y_{18}+y_8$	$y_{17}+y_9$	$y_{16}+y_{10}$	$y_{15}+y_{11}$	$y_{14}+y_{12}$	y_{13}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	25	a_{01}
a_{02}	−253	−138	−33	62	147	222	287	342	387	422	447	462	467	5175	a_{03}
a_{22}	92	69	48	29	12	−3	−16	−27	−36	−43	−48	−51	−52	53820	a_{23}
a_{04}	1265	−345	−1122	−1255	−915	−255	590	1503	2385	3155	3750	4125	4253	30015	a_{05}
a_{24}	−143198	10373	99385	137803	138262	112067	69193	18285	−33342	−79703	−116143	−139337	−147290	34337160	a_{25}
a_{44}	1518	253	−517	−897	−982	−857	−597	−267	78	393	643	803	858	34337160	a_{45}

	$y_{25}-y_1$	$y_{24}-y_2$	$y_{23}-y_3$	$y_{22}-y_4$	$y_{21}-y_5$	$y_{20}-y_6$	$y_{19}-y_7$	$y_{18}-y_8$	$y_{17}-y_9$	$y_{16}-y_{10}$	$y_{15}-y_{11}$	$y_{14}-y_{12}$	Denominator	
a_{11}	12	11	10	9	8	7	6	5	4	3	2	1	1300	a_{12}
a_{13}	−30866	−8602	8525	20982	29236	33754	35003	33450	29562	23806	16649	8558	1776060	a_{14}
a_{33}	506	253	55	−93	−196	−259	−287	−285	−258	−211	−149	−77	1776060	a_{34}
a_{15}	8322182	−6024183	−9604353	−6671883	−544668	6301491	12139321	15896511	17062146	15593141	11820675	6356625	429214500	—
a_{35}	−284372	144463	293128	266403	146408	−5131	−144616	−244311	−290076	−279101	−217640	−118745	343371600	—
a_{55}	1012	−253	−748	−753	−488	−119	236	501	636	631	500	275	156078000	—

$n=26$

	$y_{26}+y_1$	$y_{25}+y_2$	$y_{24}+y_3$	$y_{23}+y_4$	$y_{22}+y_5$	$y_{21}+y_6$	$y_{20}+y_7$	$y_{19}+y_8$	$y_{18}+y_9$	$y_{17}+y_{10}$	$y_{16}+y_{11}$	$y_{15}+y_{12}$	$y_{14}+y_{13}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	26	a_{01}
a_{02}	−138	−78	−23	27	72	112	147	177	202	222	237	247	252	2912	a_{03}
a_{22}	50	38	27	17	8	0	−7	−13	−18	−22	−25	−27	−28	32760	a_{23}
a_{04}	1518	−330	−1265	−1485	−1170	−482	435	1455	2470	3390	4143	4675	4950	36608	a_{05}
a_{24}	−518650	13662	331683	479743	498142	423150	287007	117923	−59922	−226378	−365325	−464673	−516362	137348640	a_{25}
a_{44}	2530	506	−759	−1419	−1614	−1470	−1099	−599	−54	466	905	1221	1386	68674320	a_{45}

	$y_{26}-y_1$	$y_{25}-y_2$	$y_{24}-y_3$	$y_{23}-y_4$	$y_{22}-y_5$	$y_{21}-y_6$	$y_{20}-y_7$	$y_{19}-y_8$	$y_{18}-y_9$	$y_{17}-y_{10}$	$y_{16}-y_{11}$	$y_{15}-y_{12}$	$y_{14}-y_{13}$	Denominator	
a_{11}	25	23	21	19	17	15	13	11	9	7	5	3	1	2925	a_{12}
a_{13}	−304750	−94438	69391	190779	273768	322400	340717	332761	302574	254198	191675	119047	40356	18729360	a_{14}
a_{33}	1150	598	161	−171	−408	−560	−637	−649	−606	−518	−395	−247	−84	4682340	a_{34}
a_{15}	−143088770	−92843591	−159650659	−121174609	−27852494	82001662	180920753	251008483	282663258	273302078	226084429	148636175	51778150	7741468800	—
a_{35}	−140530	62330	138299	132753	81702	12682	−54353	−106139	−135210	−139006	−118981	−79711	−28002	193536720	—
a_{55}	2530	−506	−1771	−1881	−1326	−482	377	1067	1482	1582	1381	935	330	483841800	—

$n=27$

	$y_{27}+y_1$	$y_{26}+y_2$	$y_{25}+y_3$	$y_{24}+y_4$	$y_{23}+y_5$	$y_{22}+y_6$	$y_{21}+y_7$	$y_{20}+y_8$	$y_{19}+y_9$	$y_{18}+y_{10}$	$y_{17}+y_{11}$	$y_{16}+y_{12}$	$y_{15}+y_{13}$	y_{14}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	27	a_{01}
a_{02}	−60	−35	−12	9	28	45	60	73	84	93	100	105	108	109	1305	a_{03}
a_{22}	325	250	181	118	61	10	−35	−74	−107	−134	−155	−170	−179	−182	237510	a_{23}
a_{04}	37950	−6325	−29670	−36432	−30580	−15705	4980	28640	52818	75435	94790	109560	118800	121943	930465	a_{05}
a_{24}	−331890	−5290	195178	294515	315548	278930	203140	104483	−2910	−107082	−198250	−268805	−313312	−328510	96768360	a_{25}
a_{44}	2990	690	−782	−1587	−1872	−1770	−1400	−867	−262	338	870	1285	1548	1638	96768360	a_{45}

	$y_{27}-y_1$	$y_{26}-y_2$	$y_{25}-y_3$	$y_{24}-y_4$	$y_{23}-y_5$	$y_{22}-y_6$	$y_{21}-y_7$	$y_{20}-y_8$	$y_{19}-y_9$	$y_{18}-y_{10}$	$y_{17}-y_{11}$	$y_{16}-y_{12}$	$y_{15}-y_{13}$	Denominator	
a_{11}	13	12	11	10	9	8	7	6	5	4	3	2	1	1638	a_{12}
a_{13}	65260	22090	−11924	−37545	−55536	−66660	−71680	−71359	−66460	−57746	−45980	−31925	−16344	4275180	a_{14}
a_{33}	130	70	22	−15	−42	−60	−70	−73	−70	−62	−50	−35	−18	610740	a_{34}
a_{15}	17059445	−9897130	−18358439	−15007684	−5186694	6982132	18473882	27239188	32082283	32539058	28755119	21363844	11364440	967683600	—
a_{35}	−494845	190210	464255	468096	314742	91932	−135338	−322288	−441827	−482026	−443591	−337336	−181656	774146880	—
a_{55}	16445	−2530	−10879	−12144	−9174	−4188	1162	5728	8803	10058	9479	7304	3960	3870734400	—

$n = 28$

	$y_{28}+y_1$	$y_{27}+y_2$	$y_{26}+y_3$	$y_{25}+y_4$	$y_{24}+y_5$	$y_{23}+y_6$	$y_{22}+y_7$	$y_{21}+y_8$	$y_{20}+y_9$	$y_{19}+y_{10}$	$y_{18}+y_{11}$	$y_{17}+y_{12}$	$y_{16}+y_{13}$	$y_{15}+y_{14}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	28	a_{01}
a_{02}	−65	−39	−15	7	27	45	61	75	87	97	105	111	115	117	1456	a_{03}
a_{22}	117	91	67	45	25	7	−9	−23	−35	−45	−53	−59	−63	−65	95004	a_{23}
a_{04}	7475	−897	−5475	−7015	−6210	−3690	−22	4290	8805	13145	16995	20103	22280	23400	186368	a_{05}
a_{24}	−420615	−23075	226935	357483	394298	360770	277950	164550	36943	−90837	−207095	−302475	−369960	−404872	134634240	a_{25}
a_{44}	1755	455	−395	−879	−1074	−1050	−870	−590	−259	81	395	655	840	936	67317120	a_{45}

	$y_{28}-y_1$	$y_{27}-y_2$	$y_{26}-y_3$	$y_{25}-y_4$	$y_{24}-y_5$	$y_{23}-y_6$	$y_{22}-y_7$	$y_{21}-y_8$	$y_{20}-y_9$	$y_{19}-y_{10}$	$y_{18}-y_{11}$	$y_{17}-y_{12}$	$y_{16}-y_{13}$	$y_{15}-y_{14}$	Denominator	
a_{11}	27	25	23	21	19	17	15	13	11	9	7	5	3	1	3654	a_{12}
a_{13}	−1267695	−462475	178595	668647	1020813	1248225	1364015	1381315	1313257	1172973	973595	728255	450085	152217	88353720	a_{14}
a_{33}	585	325	115	−49	−171	−255	−305	−325	−319	−291	−245	−185	−115	−39	3155490	a_{34}
a_{15}	1042076295	−538691855	−1079419555	−940534469	−413158154	274867322	952952762	1501735222	1847051393	1953910984	1820470099	1472004629	951883621	350542680	61931750400	—
a_{35}	−874575	288535	781195	823837	595194	235926	−147754	−480454	−713977	−823695	−804923	−669293	−441128	−153816	1548293760	—
a_{55}	13455	−1495	−8395	−9821	−7866	−4182	−22	3718	6457	7887	7931	6701	4456	1560	3870734400	—

$n = 29$

	$y_{29}+y_1$	$y_{28}+y_2$	$y_{27}+y_3$	$y_{26}+y_4$	$y_{25}+y_5$	$y_{24}+y_6$	$y_{23}+y_7$	$y_{22}+y_8$	$y_{21}+y_9$	$y_{20}+y_{10}$	$y_{19}+y_{11}$	$y_{18}+y_{12}$	$y_{17}+y_{13}$	$y_{16}+y_{14}$	y_{15}	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	29	a_{01}
a_{02}	−351	−216	−91	24	129	224	309	384	449	504	549	584	609	624	629	8091	a_{03}
o_{22}	126	99	74	51	30	11	−6	−21	−34	−45	−54	−61	−66	−69	−70	113274	a_{23}
a_{04}	17550	−1350	−12025	−16050	−14874	−9820	−2085	7260	17270	27126	36135	43730	49470	53040	54251	445005	a_{05}
a_{24}	−105651	−9594	52156	85874	97333	91804	74056	48356	18469	−12342	−41316	−66194	−85219	−97136	−101192	37024416	a_{25}
a_{44}	4095	1170	−780	−1930	−2441	−2460	−2120	−1540	−825	−66	660	1200	1775	2080	2184	185122080	a_{45}

	$y_{29}-y_1$	$y_{28}-y_2$	$y_{27}-y_3$	$y_{26}-y_4$	$y_{25}-y_5$	$y_{24}-y_6$	$y_{23}-y_7$	$y_{22}-y_8$	$y_{21}-y_9$	$y_{20}-y_{10}$	$y_{19}-y_{11}$	$y_{18}-y_{12}$	$y_{17}-y_{13}$	$y_{16}-y_{14}$	Denominator	
a_{11}	14	13	12	11	10	9	8	7	6	5	4	3	2	1	2030	a_{12}
a_{13}	477477	−185796	48646	230252	363425	452568	502084	516376	499847	456900	391938	309364	213581	108992	35341488	a_{14}
a_{33}	819	468	182	−44	−215	−336	−412	−448	−449	−420	−366	−292	−203	−104	5048784	a_{34}
a_{15}	11494665	−5271435	−11441560	−10543335	−5453373	1548474	8724864	14826714	19038949	20926251	20378808	17558063	12842463	6773208	715244400	—
a_{35}	−286650	80145	243490	267685	205354	98418	−20932	−129262	−210922	−257073	−264714	−235709	−175814	−93704	572195520	
o_{55}	8190	−585	−4810	−5885	−4958	−2946	−556	1694	3454	4521	4818	4373	3298	1768	2860977600	—

5—1387P—IV

$n = 30$

	$y_{30}+y_1$	$y_{29}+y_2$	$y_{28}+y_3$	$y_{27}+y_4$	$y_{26}+y_5$	$y_{25}+y_6$	$y_{24}+y_7$	$y_{23}+y_8$	$y_{22}+y_9$	$y_{21}+y_{10}$	$y_{20}+y_{11}$	$y_{19}+y_{12}$	$y_{18}+y_{13}$	$y_{17}+y_{14}$	$y_{16}+y_{15}$	Denominator	
a_{00}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	30	a_{01}
a_{02}	−189	−119	−54	6	61	111	156	196	231	261	286	306	321	331	336	4480	a_{03}
a_{22}	203	161	122	86	53	23	−4	−28	−49	−67	−82	−94	−103	−109	−112	201376	a_{23}
a_{04}	8775	−325	−5616	−7800	−7525	−5385	−1920	2384	7095	11835	16280	20160	23259	25415	26520	226304	a_{05}
a_{24}	−6579027	−816543	2962440	5107232	5940273	5757133	4826512	3390240	1663277	−166287	−1937232	−3515208	−4792735	−5689203	−6150872	2517660288	a_{25}
a_{44}	23751	7371	−3744	−10504	−13749	−14249	−12704	−9744	−5929	−1749	2376	6096	9131	11271	12376	1258830144	a_{45}

	$y_{30}-y_1$	$y_{29}-y_2$	$y_{28}-y_3$	$y_{27}-y_4$	$y_{26}-y_5$	$y_{25}-y_6$	$y_{24}-y_7$	$y_{23}-y_8$	$y_{22}-y_9$	$y_{21}-y_{10}$	$y_{20}-y_{11}$	$y_{19}-y_{12}$	$y_{18}-y_{13}$	$y_{17}-y_{14}$	$y_{16}-y_{15}$	Denominator	
a_{11}	29	27	25	23	21	19	17	15	13	11	9	7	5	3	1	4495	a_{12}
a_{13}	−653283	−268911	42750	287086	469483	595327	670004	698900	687401	640893	564762	464394	345175	212491	71728	51264576	a_{14}
a_{33}	1827	1071	450	−46	−427	−703	−884	−980	−1001	−957	−858	−714	−535	−331	−112	12816144	a_{34}
a_{15}	19336350735	−7823192715	−18469189440	−17891310840	−10440817765	394366641	11914584096	22133368336	29702372271	33836295141	34237809672	31022489232	24643734987	15817703057	5448231672	1258830144000	—
a_{35}	−14030055	3268895	11320920	12940720	10449845	5718487	201272	−5008448	−9169303	−11822613	−12768096	−12021576	−9775691	−6360601	−2204696	31470753600	—
a_{55}	16965	−585	−9360	−11960	−10535	−6821	−2176	2384	6149	8679	9768	9408	7753	5083	1768	7152444000	—

$n > 30$

$$\eta_0 = \left[(y_n + y_1) + (y_{n-1} + y_2) + \quad \ldots \right] \qquad a_{00} = \frac{1}{n}\eta_0 = a_{01}$$

$$\eta_1 = \left[\left(\frac{n-1}{2}\right)(y_n - y_1) + \left(\frac{n-3}{2}\right)(y_{n-1} - y_2) + \quad \ldots \right] \qquad a_{11} = \frac{12}{n(n^2-1)}\eta_1 = a_{12}$$

$$\eta_2 = \left[\left(\frac{n-1}{2}\right)^2 (y_n + y_1) + \left(\frac{n-3}{2}\right)^2 (y_{n-1} + y_2) + \quad \ldots \right] \qquad a_{02} = \frac{3(3n^2-7)}{4n(n^2-4)}\eta_0 - \frac{15}{n(n^2-4)}\eta_2 = a_{03}$$

$$\eta_3 = \left[\left(\frac{n-1}{2}\right)^3 (y_n - y_1) + \left(\frac{n-3}{2}\right)^3 (y_{n-1} - y_2) + \quad \ldots \right] \qquad a_{22} = \frac{-15}{n(n^2-4)}\eta_0 + \frac{180}{n(n^2-1)(n^2-4)}\eta_2 = a_{23}$$

$$\eta_4 = \left[\left(\frac{n-1}{2}\right)^4 (y_n + y_1) + \left(\frac{n-3}{2}\right)^4 (y_{n-1} + y_2) + \quad \ldots \right] \qquad a_{13} = \frac{25(3n^4 - 18n^2 + 31)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_1 - \frac{140(3n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_3 = a_{14}$$

$$\eta_5 = \left[\left(\frac{n-1}{2}\right)^5 (y_n - y_1) + \left(\frac{n-3}{2}\right)^5 (y_{n-1} - y_2) + \quad \ldots \right] \qquad a_{33} = \frac{-140(3n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)}\eta_1 + \frac{2800}{n(n^2-1)(n^2-4)(n^2-9)}\eta_3 = a_{34}$$

$$a_{04}=\frac{15(15n^4-230n^2+407)}{64n(n^2-4)(n^2-16)}\eta_0-\frac{525(n^2-7)}{8n(n^2-4)(n^2-16)}\eta_2+\frac{945}{4n(n^2-4)(n^2-16)}\eta_4=a_{05}$$

$$a_{24}=\frac{-525(n^2-7)}{8n(n^2-4)(n^2-16)}\eta_0+\frac{2205(n^4-10n^2+29)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_2-\frac{3150(3n^2-13)}{n(n^2-1)(n^2-4)(n^2-16)}\eta_4=a_{25}$$

$$a_{44}=\frac{945}{4n(n^2-4)(n^2-16)}\eta_0-\frac{3150(3n^2-13)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_2+\frac{44100}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)}\eta_4=a_{45}$$

$$a_{15}=\frac{147(25n^8-900n^6+10230n^4-37060n^2+46137)}{16n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1-\frac{(3n^6-75n^4+541n^2-853)2205}{2n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2=25)}\eta_3$$

$$+\frac{693(15n^4-230n^2+407)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5$$

$$a_{35}=\frac{-2205(3n^6-75n^4+541n^2-853)}{2n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1+\frac{18900(3n^4-46n^2+199)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_3$$

$$-\frac{194040(n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5$$

$$a_{55}=\frac{693(15n^4-230n^2+407)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_1-\frac{194040(n^2-7)}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_3$$

$$+\frac{698544}{n(n^2-1)(n^2-4)(n^2-9)(n^2-16)(n^2-25)}\eta_5$$

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